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Relevancy Measures of Errors in Displacements of Finite Elements Analysis Results

A.B. Bolkhir^{*}, T. K. Yousif*Civil Engineering Department, Benghazi University, Benghazi, Libya*^{*} Correspondence: belkhir51@yahoo.com**PUBLISHED IN**

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ABSTRACT

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RELEVANCY MEASURES OF ERRORS IN DISPLACEMENTS OF FINITE ELEMENTS ANALYSIS RESULTS

A.B. Bolkhir* & T. K. Yousif

Civil Engineering Dep., Benghazi University, Benghazi, Libya

[*belkhir51@yahoo.com](mailto:belkhir51@yahoo.com)

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Abstract

This paper highlights the methods of error estimation in finite element analysis (FEA) results. It indicates that the modeling error could be eliminated either by performing finite element analysis with successively finer meshes or by extrapolating response predictions from an orderly sequence of relatively low degree of freedom analysis results. In addition, the paper eliminates the round-off error by running the code at a higher precision. The paper provides application of the procedure of error estimation in finite element analysis results. It draws a conclusion based on results of application of methods of error estimation. And recommends an analysis process that leads to accurate estimates of displacement error in finite element analysis result to measure the accuracy and a possible improvement on it by monotonic convergence of the displacements results.

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1. Introduction

The inaccuracy induced by using a finite number of generalized nodal displacements in finite element analysis is called modeling error. Our objective is to establish a reliable error estimate in displacements of a structure analyzed by finite element method. The modeling error can be classified into two categories. The first category of modeling error estimator is based on estimates of the exact solution to the differential equation which are higher-order accurate than the underlying numerical solutions and includes approaches such as Richardson extrapolation, order refinement and recovery methods from finite elements. The second category of error estimator based on the residual (i.e. the truncation error) and includes discretization error, transport equations, finite element residual methods and adjoining method extensions. There are certain restrictions on the error estimators in order to reach to a reliable error estimate. These restrictions includes mesh refinement policy rules of extrapolation that guarantee monotonic convergence of the results and use of high precision code.

2. Discretization error

The basis of FEA is taking a system governed by differential equations and partitioning it into regions (elements) that can individually be solved with simple linear equations. This partitioning creates a series of linear approximations that strive to map, as closely as possible, the true continuous solution. Using structural stress-strain as the primary example, the FEA code takes the mesh approximation and solves for the displacements at each node given the loading and constraints on the model. Next, as a secondary operation, the code approximates the stress contour in each element by looking at the relative displacement of the nodes of that element. In this manner a stress contour is determined for each element – an approximation of the stress throughout each individual element, including at the edge of the elements and the node locations. Given that this contour is approximated based only on the displacement of the nodes of the element in question, the stress contour will not necessarily be continuous from one element to the next. Essentially, this discontinuity of the stress contour from

one element to another is the discretization error. As the size of each element is reduced towards zero this error goes to zero given that the increasing number of partitions is approaching the true continuous system [7]. The number of d.o.f. is infinite in the mathematical model but finite in an FE model. The FEA solution is influenced by the number of elements used, the number of nodes per element, the nature of element shape functions. Integration rules used with isoperimetric elements and other formulation details of particular elements. [12].

3. Round-off error

Rounding error refers to loss of information due to truncation or rounding of numbers to fit a finite computer word length. This kind of error is present in element and structure matrices before a solution algorithm is applied. Due to finiteness of the computer word lengths, only a finite subset of the infinitely many real numbers can be represented in the machines. None of the irrational numbers and very few of the rational numbers can be represented [20]. The study of round-off error is crucial when solving a linear system of equations, since it may pollute and even destroy the solution of the problem. Round-off error is of particular concern in adaptive hp-finite element (FE) computations, where h denotes the element size and p the polynomial order of approximation, both varying locally throughout the computational grid. There is a potential for large round-off errors when using heavy mesh-refinements in a hp-adaptive FE discretization to approximate a particular feature of the solution. Since the condition number controls the round-off error, a natural approach to minimize error is by decreasing the condition number. One way to achieve this is by performing an optimal scaling of basic functions [6]. And the Computer round-off errors develop with the representation of floating point

numbers on the computer and the accuracy at which numbers are stored. With advanced computer resources, numbers are typically stored with 16, 32, or 64 bits. Round-off errors are not considered significant when compared with other errors. If computer round-off errors are suspected of being significant, one test is to run the code at a higher precision or on a computer known to store floating point numbers at a higher precision. One can attempt to iterate a coarse grid solution to a residual of machine zero; however, this may not be possible for more algorithms that are complex. [9]. For most minicomputers, single precision refers to a 32-bit and double precision to a 64-bit word length for real numbers which corresponds to a decimal number of between six and seven significant figures in 32-bit and ten to fifteen significant figures in 64-bit. Cases can occur when the nature of the problem and its numeric representation can produce such a build-up of round-off errors that the results produced may be quite meaningless. It may be necessary then to use double-precision arithmetic in the sensitive accuracy structural analysis programs. But the consequences, while they will include a far more accurate result, will entail penalties of increased running time and memory requirements. As we can see the case study results in single-precision (32bit) results and the double-precision (64bit) results later, the precision error more affective when the analysis results are in relative small values than the big values of the results.

4. Accuracy of mesh refinement

The creation of meshing in finite element models is the most important step in the entire analysis since the decision made at this point will affect the accuracy and the economy of the solid model. Thus, the choice of mesh for various analyses is crucial. An element or mesh that is good in one problem area such as magnetic field may be poor in another such as stress

analysis. Even in a specific problem area, an element may behave well or badly, depending on particular geometry, loading and boundary conditions, for reasons of better accuracy and efficiency, quadrilateral elements are preferred for two-dimensional meshes and hexahedral elements for three-dimensional meshes [13]. This preference is clear in structural analysis and seems to also hold for other engineering disciplines. The mesh quality has been defined as a combination of two criteria such as element quality and enough number of elements in the mesh. The quality of the surface mesh should be good to get acceptable finite element results. Element quality for quadrilateral element is defined by two factors namely: Distortion factor and Aspect ratio. When generating a finite element mesh, one can expect more accurate results with more refined mesh (smaller size but larger number of elements within a confined area). However as the model gets larger In FEA, larger model does not mean larger geometry, but rather the complexity due to the number of elements used), the computer will spend more time to generate the results of the analysis. It is often very important to minimize the computing time without a significant loss in the accuracy of the solution to achieve satisfying results with minimum computing time [19]. The most efficient element model to use for particular analysis varies with accuracy requirements, number of degree of freedom and computer implementation. Usually higher degree elements should be selected when accuracy requirements are high. The most efficient grid arrangement varies with the analysis purpose, computer code and accuracy required. Generally, optimized grid should be used for studies of elements characteristics, focused grids for production analysis of complex systems, and regular grid when extrapolation can be exploited. Use of successive grids which lack the discipline of an optimum, focused or regular arrangement can lead to an acceptable

mesh design. However, it could also incur exorbitant time penalties to attain the required accuracy or lead to oscillatory or divergent sequence of analysis results [15]. In addition to the requirement that the order of accuracy of a solver be verified, Richardson extrapolation also depends on a uniform refinement of the discretization size. When applying Richardson extrapolation to a relatively simple approximation such as that of an extrapolation to partial differential equations in one dimension, uniform refinement remains relatively simple. The computational domain, or mesh, in one dimension consists of a straight line subdivided into appropriate intervals. A typical mesh refinement consists of dividing each subinterval of the original mesh in half. The result, obviously, is a new mesh with twice as many intervals as the original, a refinement ratio of 2. In two dimensions, refinement of a structured mesh remains essentially the same, but extended to an extra dimension. The same refinement ratio of 2 increases the number of elements by a factor of 4. The issue of refinement becomes significantly more difficult, however, when dealing with unstructured meshes in two dimensions. For a sequence of meshes to be appropriate for Richardson extrapolation, the meshes must be geometrically similar. [2]

5. Accuracy in extrapolation

Richardson [4] bases the most commonly used extrapolation-based discretization error estimator off work in 1910. The method was used to calculate a fourth-order accurate solution for a second-order finite-difference calculation of stresses in a masonry dam. The method was applied to solutions obtained on multiple mesh levels with grid doubling to estimate the true asymptotic solution Richardson [7]. Later explores implementations and pitfalls of the method and demonstrates accuracy with simple applications such as the calculation of π and the natural log base e . Richardson used a refinement factor of two

(i.e. grid doubling) and a formal order of accuracy of two, Richardson extrapolation has since been generalized to apply to any formal order of accuracy and refinement factor [8].

Richardson extrapolation makes the assumption that the solution is smooth and that the two solutions are asymptotic. Extrapolation can in some cases increase oscillations in the solution and can magnify iterative and round-off error. In addition, the extrapolated values are generally not conservative [8] and can result in a less accurate solution than the fine grid solution. With these considerations, Richardson extrapolation was eventually adapted as an error estimator expressed as the difference between the numerical solution and the extrapolated solution forming the base for the error and uncertainty estimators included in this study. The success of Richardson extrapolation is due to its generality because it can be applied to any scientific computation regardless of the discretization method (e.g. finite-difference, finite-volume, or finite-element) and because it is an a-posteriori error estimator, the method is not code intrusive and relatively easy to implement. Extrapolation is a methodology for improving the order of accuracy of numerical solutions that involve the use of a discretization size h . By combining the results from numerical solutions using a sequence of related discretization sizes, the leading order error terms can be methodically removed, resulting in higher order accurate results. Richardson extrapolation is commonly used within the numerical approximation of partial differential equations to improve certain predictive quantities such as structures deformation, once these quantities are calculated on a sequence of meshes. Within this paper, Richardson extrapolation is applied directly to the solution algorithm used within existing numerical solvers of partial differential

equations to increase the order of accuracy of the numerical result without referring to the details of the methodology or its implementation within the numerical code. Only the order of accuracy of the existing solver and certain interpolations required to pass information between the mesh levels are needed to improve the order of accuracy and the overall solution accuracy. Richardson extrapolation relies on an observation about the shape of the error terms in a numerical approximation. Assume $N(h)$ is a numerical approximation of order p to an exact result $N(0)$.

The algorithm is consistent when the exact result is obtained as h goes to zero. Richardson extrapolation has been used to increase the order of accuracy of numerical approximations to the derivative of a function and to the integral of a function over a bounded domain. It can also be applied to the numerical solution of ordinary differential equations, although the computational cost is greater than certain specialized methods. Unfortunately, the application of Richardson extrapolation to partial differential equations is subject to certain issues that must be addressed for the method to achieve higher order accuracy. These issues include the effects of dispersion, which can cause variation in the locations of physical features of the solution based on mesh resolution, improper implementation of the algorithm so that the original theoretical order of accuracy is not achieved, and poor mesh refinement resulting in sequences of meshes that are not geometrically similar. Several issues regarding Richardson extrapolation have been addressed. Successful application to partial differential equations that exhibit significant dispersion has been performed via application to the residual rather than the solution. The issue of proper code implementation has been addressed through extensive code verification, either through comparison to the known exact

solution, or through a "manufactured" solution. Lastly, geometrically similar meshes have been created through h-refinement. [1,2,21,22]

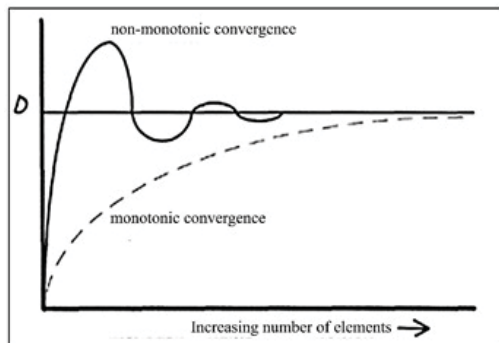


Figure 1. Convergence, Reference: steven lepi, "A Practical Guide to finite elements a solid mechanics approach", page 198

6. Monotonic convergence

To illustrate the concept of monotonic convergence, the graph in Figure.1 shows exact displacement, D , at some point in a structure. The dashed monotonic convergence line suggests that as the finite element mesh is refined, the finite element solution approaches (converges upon) the exact value. Note that for every increase in the number of elements, displacement on the monotonic line becomes a better approximation for the exact displacement. When convergence is monotonic, the analyst knows that each successive refinement yields a more accurate solution than the preceding one. The analyst can therefore intuit if further mesh refinement will yield significantly better results, or if the current level of refinement is sufficient. However, the same cannot be said when convergence is non-monotonic. When convergence is non-monotonic, further mesh refinement may provide better or worse accuracy, making it difficult to determine the number of elements needed in a particular model. Monotonic convergence has an advantage in that the degree of mesh refinement needed for a particular level of accuracy can be inferred by observing how much the solution changes between each refinement. For example, if the change in displacement is small with each successive mesh

refinement, the solution is assumed to have converged, and further refinement will not render a more accurate model. (This, of course, does not always mean that the correct answer has been obtained, only that the results will not be more accurate with further refinement.) In light of the advantages of monotonic convergence, finite elements are typically designed to ensure such behavior, although there are some exceptions which will be mentioned shortly. To ensure monotonic convergence of the finite element solution, both the individual elements and the assemblage of elements ("the mesh") must meet certain requirements. The term h-convergence is used in reference to the process where the predicted displacement in a finite element model had better represent the exact displacement as a larger number of smaller elements is employed. Only h-type elements are considered in this text, thus, the term monotonic convergence herein implies convergence using the h-method of finite elements.

Displacement within a finite element is interpolated from the discrete values of displacement at the element's nodes. To generate a finite element interpolation function, one usually begins with a displacement assumption, typically a polynomial with adjustable terms (a_i). That a displacement assumption must:

- 1- Have sufficiently differentiable terms
- 2- Be complete
- 3- Have linearly independent terms
- 4- Satisfy the essential boundary conditions of the element

If a displacement assumption used for simple one-dimensional problems satisfies the requirements above, the finite element model will exhibit desirable convergence characteristics, namely, monotonic convergence.

Requirements to ensure monotonic convergence:

To ensure monotonic convergence of displacement predicted by the finite element method, both the individual

elements and the assemblage of elements ("the mesh") should meet the five requirements described below although the property of geometric isotropy is not included, it will be considered a general requirement for displacement assumptions. Requirements 1 and 2 are equivalent to maintaining completeness in energy, while Requirement 3 ensures that inter-element compatibility is maintained.

- 1- Rigid body representation: The displacement assumption must be able to account for all rigid body displacement modes of the element.
- 2- Uniform Strain representation: Constant strain states for all strain components specified in the constitutive equations of a particular idealization must be represented within the element as the largest dimension of the element approaches zero.
- 3- Compatibility between Elements: The dependent variable(s), and P1 derivatives of the dependent variable, must be continuous at the nodes and across the inter-element boundaries of adjacent elements.
- 4- Mesh refinement: Each successive mesh refinement must contain all of the previous nodes and elements in their original location
- 5- Uniform strain representation: The mesh must be able to represent uniform strain when boundary conditions that are consistent with a uniform strain condition are imposed

If all of the requirements are met, the finite element solution will converge, monotonically, with mesh refinement. This presumes that other errors, such as computational round off error, are not introduced in the solution procedure. Two conditions are required for such monotonic convergence of finite element solution: the elements must be complete and the element and mesh must be compatible.

7. Illustrative case study

The straight cantilever beam (see Fig.4) is a frequently used test problem which can be applied to beam, plate, and solid elements, its virtues are its simplicity and the fact that all of the principal element deformation modes described earlier (constant and linearly varying strains and curvatures) can be evoked by loads applied to the free end. We have added different element intensity principally to test the accuracy of finite element analysis results due to discretization and mesh refinements, to reduce the effect of round off error in the FEA results we intend to use single (32-bit) and double (64-bit) precision digits in the analysis calculations. The design of a comprehensive set of element test problems should, of course, take into account the parameters, which affect accuracy. These parameters can be classified under the headings of loading, element geometry, problem geometry, and material properties. With regard to loading, the problem set should, as a whole, provide significant loading for each of the types of deformation, which the elements can exhibit. Each element that discrete has a standard shape which may be the only shape that the developer has tested. In all cases the standard shape is a square to be used but the discrete intensity will vary for each case element. Poisson's ratio has a strong effect on element accuracy as its value approaches 0.5. Such values should be included in the problem set if the use of nearly incompressible materials is contemplated. Plasticity affects element accuracy in much the same way as incompressible material. Plasticity and all other nonlinear effects are outside the scope of the paper. Anisotropic material properties also have a significant effect on element accuracy, which will not be examined here. The test element parameters Straight cantilever beam elements. Length = 6.0 in; height = 0.2 in; thickness = 0.1 in; modulus of elasticity $E = 1.0 \times 10^7$ lb./in²; Poisson's ratio $\nu = 0.30$; modulus of rigidity $G = 3,846,154$ lb.

/in2. Loading= one unit forces at free end upward. (Note: All elements have equal volume and QUAD 4 type.) The incompatible bending modes option should be applied to prevent the Poisson's effect extra stress at the support. [13]. The independent solution is derived using elementary beam theory that assumes no local Poisson's effect occurs at the support. The beam is modeled in commercial software SAP2000 to match this assumptions.

As we can see in Table I lists the extrapolated tip deformation by Richardson polynomial extrapolation associated finite element analysis using h^0 , h^2 , h^4 , h^6 function. These estimates agree with the actual tip deformation to the fifteen digits of accuracy. As we can see the results of the tip deformation values found that the first three polynomials close to the theoretical calculated value and by closer look we conclude that the third extrapolation polynomial is the most accurate value to the theoretical calculated value. Therefore, the extrapolation equation will be:

$$\Delta = 0.10807800014558 \times h^0 - 2.70000002825901 \times 10^{-2} \times h^2 + 9.42123920642551 \times 10^{-10} \times h^4 + 1.13479955757952 \times 10^{-9} \times h^6$$

Table 1. Extrapolated Beam Tip Deformation at Each Element Size

No of elements	element size (1/No. elements)	Extrapolated Beam Tip deformation(in)
1	1.000000000000000	0.081078001808891
2	0.500000000000000	0.10132800020524
4	0.250000000000000	0.106390500000853
8	0.125000000000000	0.107656125010377
16	0.062500000000000	0.107972531263469
32	0.031250000000000	0.108051632786849

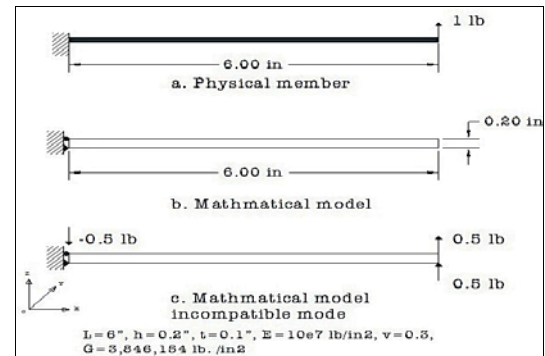


Figure 2. a. Beam element physical model including applied load. b. mathematical model. c. mathematical model in incompatible bending mode

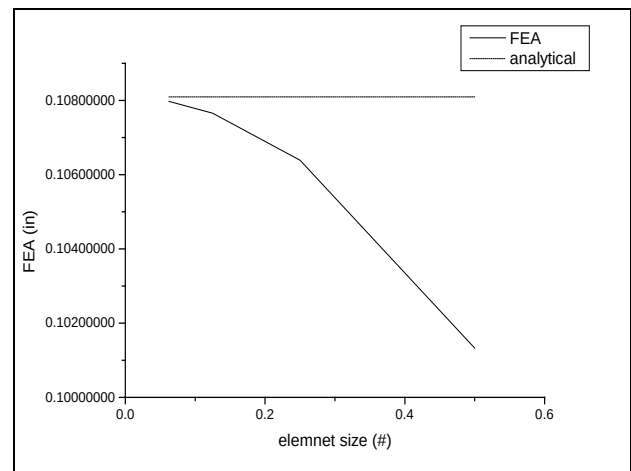


Figure 3. the case study monotonicity proof.

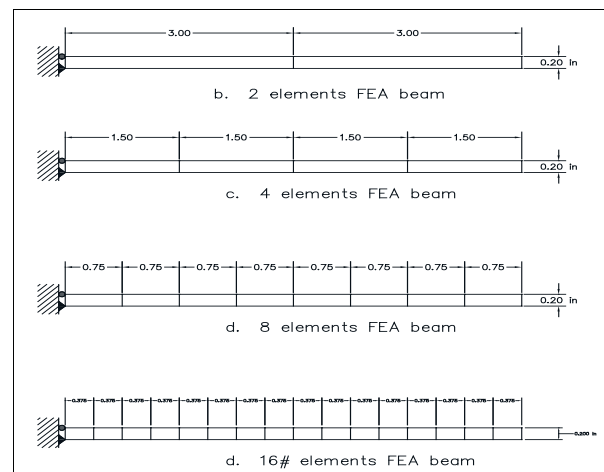


Figure 4. FEA Beam mesh refinement.

Table 2. Mesh Refinement Results of Deformation at Free End

No. of Elements	Single precision (32 bit)	Double Precision (64bit)	Difference =(64bit-32bit)	error % = (difference / DP64bit)*100
1	0.0810780000124783	0.0810780000124783	0.000000000000000E+00	0.00000000000000000000000000000000%
2	0.1013280000204860	0.1013280000204880	1.99840144432528E-15	0.00000000000197221048863218%
4	0.1063905000008500	0.1063905000008500	0.000000000000000E+00	0.00000000000000000000000000000000%
8	0.1076561250103760	0.1076561250103760	0.000000000000000E+00	0.00000000000000000000000000000000%
16	0.1079725312634680	0.1079725312634450	-2.29954943975486E-14	-0.00000000002129754126208380%
32	0.1080516327874710	0.1080516327868490	-6.22002449546244E-13	-0.00000000057565298506247800%

8. Discussion and Conclusion

Table 3. Results

method	Tip deformation	% of error
Analytical Tip deformation	0.108093599996256	0.00000%
Mesh Refinement	0.108051632786849	-0.03883%
Extrapolated Tip deformation	0.108078000014558	-0.01443%

This paper propose the procedure required to obtain a reliable estimate of error in displacement of finite element analysis results. The following conclusion may be drawn:

- 1- the Richardson's extrapolation technique may be used to obtain a reliable measure of error in displacement of finite element analysis results if certain conditions are satisfied such as:
 - a. high precision code is used in the analysis (64-Bit) system.
 - b. Use of polynomial error functions in the form ($\Delta = a_0 + a_1 * h^2 + a_2 * h^4 + a_3 * h^6$)
 - c. Monotonic convergence of the displacement function is guaranteed
 - d. The results of finite element are obtained by mean of the confirming elements.
- 2- The case study shows that the error estimate in displacement at the free end of the cantilever using extrapolation is (-0.01443) percent in comparison with analytical results. It also shows that the error estimate in displacement at the free end of the cantilever free end using mesh refinement is(-0.03882) percent in comparison with the analytical results

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